Topic 11b: Probability: Binomial Distribution

The Binomial distribution refers to a whole class of discrete distributions. The Binomial distribution occurs in cases where

1. **We have a fixed number of trials (i.e., attempts) are done (run n trials).**
2. **There are only 2 possible outcomes of each trial: one is called a success, the other called a failure.**
3. **The probability of success is the same for each trial. We designate the probability of success on any one trial as p. This means that the probability of a failure is (1-p).**
4. **The trials are independent; knowing the outcome of one trial tells you nothing about the outcome of the next trial.**
5. **We are looking at a random variable, X, that is the number of successes in those n trials.**

If we are looking at 6 trials and we want to find 2 successes, then there are many ways this could happen. For example, we could get SSFFFF or SFFSFF or SFFFFS or FFSFSF and so on. In fact, the number of possible ways for us to get 2 successes out of 6 attempts would be the number of combinations of 6 things taken 2 at a time, i.e. $_6C_2$. Each trial has a probability of $p^2(1-p)^4$. So for our binomial distribution the probability of getting any one of the cases with 2 successes and 4 failures is $_6C_2\, p^2(1-p)^4$.

For a particular case, say spinning a coin 6 times where a success is coming up HEADS and the probability of success for that coin on any one spin is 0.573, the probability of getting exactly two successes is $_6C_2 \cdot 0.573^2 \cdot (1 - 0.573)^4$

This would not be hard for us to compute in R, as long as we either remember the factorial formula for finding the number of combinations of n things taken r at a time, or we load one of our functions to do this.

```
      #  As an example, if the probability of success is
      #  0.573 and we have 6 trials and we want 2 successes,
      #  we can get the probability of that via
source("../combinations.R")
nCr(6,2)*0.573^2*(1-0.573)^4
```

```
>       #  As an example, if the probability of success is
>       #  0.573 and we have 6 trials and we want 2 successes,
>       #  we can get the probability of that via
> source("../combinations.R")
> nCr(6,2)*0.573^2*(1-0.573)^4
[1] 0.1637239
```

In the old days, before we had calculators and computers, this computation would have been quite time consuming. Therefore, statistics books generally had tables in them to help us find the answer to such a problem. One such table, one of many in the book, might have been:

Binomial Probabilities Table: $P(X = k)$ for n=6																			
k	Choice of probability of success on single trial																		
	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
0	0.7351	0.5314	0.3771	0.2621	0.1780	0.1176	0.0754	0.0467	0.0277	0.0156	0.0083	0.0041	0.0018	0.0007	0.0002	0.0001	0.0000	0.0000	0.0000
1	0.2321	0.3543	0.3993	0.3932	0.3560	0.3025	0.2437	0.1866	0.1359	0.0938	0.0609	0.0369	0.0205	0.0102	0.0044	0.0015	0.0004	0.0001	0.0000
2	0.0305	0.0984	0.1762	0.2458	0.2966	0.3241	0.3280	0.3110	0.2780	0.2344	0.1861	0.1382	0.0951	0.0595	0.0330	0.0154	0.0055	0.0012	0.0001
3	0.0021	0.0146	0.0415	0.0819	0.1318	0.1852	0.2355	0.2765	0.3032	0.3125	0.3032	0.2765	0.2355	0.1852	0.1318	0.0819	0.0415	0.0146	0.0021
4	0.0001	0.0012	0.0055	0.0154	0.0330	0.0595	0.0951	0.1382	0.1861	0.2344	0.2780	0.3110	0.3280	0.3241	0.2966	0.2458	0.1762	0.0984	0.0305
5	0.0000	0.0001	0.0004	0.0015	0.0044	0.0102	0.0205	0.0369	0.0609	0.0938	0.1359	0.1866	0.2437	0.3025	0.3560	0.3932	0.3993	0.3543	0.2321
6	0.0000	0.0000	0.0000	0.0001	0.0002	0.0007	0.0018	0.0041	0.0083	0.0156	0.0277	0.0467	0.0754	0.1176	0.1780	0.2621	0.3771	0.5314	0.7351

In the old days we would have looked across the 2 successes row and found that if the probability of success on a single trial was 0.55 then the answer would be 0.1861. If the probability of success on a single trial was 0.60 then the answer would be 0.1382. In our problem the probability of success on a single trial was given as 0.573. Therefore, we expect our answer to be between 0.1861 and 0.1382. We would use mathematical interpolation to make a good guess at the answer. 0.573 is 46% of the way from 0.55 to 0.60 so our answer would be 0.1861 + 0.46*(0.1382-0.1861) = 0.164066, which is not far away from the correct anser, 0.1637239.

Fortunately, you will not be asked to do this.

The table given above would be most helpful in finding the probability of getting exactly 2 successes in 6 trials if the probability of success on a single trrial was 0.35. We could just look up that answer to find that it is 0.3280. We could even find the probability of getting 3 or fewer successes on 6 trials if the probability of success on a single trial is 0.45. To get that answer we would add **P(X=0) + P(X=1) + P(X=2) + P(X=3)** or **0.0277 + 0.1359 + 0.2780 + 0.3032 = 0.7448**.

That is a lot of work! And, that kind of problem, **P(X≤3)** happens often. Therefore, instead of printing tables like the one shown above, most textbooks print cumulative binomial probability tables. The one for 6 trials is:

Cumulative Binomial Probabilities Table: P(X ≤ k) for n=6																			
k	Choice of probability of success on single trial																		
	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
0	0.7351	0.5314	0.3771	0.2621	0.1780	0.1176	0.0754	0.0467	0.0277	0.0156	0.0083	0.0041	0.0018	0.0007	0.0002	0.0001	0.0000	0.0000	0.0000
1	0.9672	0.8857	0.7765	0.6554	0.5339	0.4202	0.3191	0.2333	0.1636	0.1094	0.0692	0.0410	0.0223	0.0109	0.0046	0.0016	0.0004	0.0001	0.0000
2	0.9978	0.9842	0.9527	0.9011	0.8306	0.7443	0.6471	0.5443	0.4415	0.3438	0.2553	0.1792	0.1174	0.0705	0.0376	0.0170	0.0059	0.0013	0.0001
3	0.9999	0.9987	0.9941	0.9830	0.9624	0.9295	0.8826	0.8208	0.7447	0.6563	0.5585	0.4557	0.3529	0.2557	0.1694	0.0989	0.0473	0.0158	0.0022
4	1.0000	0.9999	0.9996	0.9984	0.9954	0.9891	0.9777	0.9590	0.9308	0.8906	0.8364	0.7667	0.6809	0.5798	0.4661	0.3446	0.2235	0.1143	0.0328
5	1.0000	1.0000	1.0000	0.9999	0.9998	0.9993	0.9982	0.9959	0.9917	0.9844	0.9723	0.9533	0.9246	0.8824	0.8220	0.7379	0.6229	0.4686	0.2649
6	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

We can read the answer to our problem, for **6** trials with the probability of success on one trial being **0.45**, **P(X≤3) = 0.7447**, essentially the same value that we found above (the difference being the result of rounding values in the tables).

Having the cumulative probabilities also helps with problems such as, for the same **6** trials, but with the probability of success on a single trial being **0.70**, find **P(X≥2)**. We can do this because **P(X≥2) = 1 - P(X≤1) = 1 - 0.0109 = 0.9891**.

By the way, the web page has links to a page of tables for the straight binomial distribution and to a page for the cumulative binomial distribution.

All of this work using tables seems out of date. In fact, R has a function that completely replaces the use of the cumulative binary distrbution table. That function is **pbinom()**. We can solve the last problems, in R, using **pbinom()**.

```
      # demonstrate the use of pbinom()
      # For probability of a single success equal to 0.35
      # in 6 trials find the probability of getting 3
      # or fewer successes.
pbinom( 3, 6, 0.45)
      # Change probability of single success on one trial
      # to 0.70, in 6 trials, find P(X>=2)
1 - pbinom( 1, 6, 0.70)
```

```
>      # demonstrate the use of pbinom()
>      # For probability of a single success equal to 0.35
>      # in 6 trials find the probability of getting 3
>      # or fewer successes.
> pbinom( 3, 6, 0.45)
[1] 0.7447361
>      # Change probability of single success on one trial
>      # to 0.70, in 6 trials, find P(X>=2)
> 1 - pbinom( 1, 6, 0.70)
[1] 0.989065
```

Here is another problem. In the binomial distribution of **13** trials, when the probability, p, of success on a single trial is p=0.64, find the probability of getting 8 or fewer successes.

```
      # For 13 trials with p=0.64 find P(X<=8)
pbinom( 8, 13, 0.64 )
```

```
>      # For 13 trials with p=0.64 find P(X<=8)
> pbinom( 8, 13, 0.64 )
[1] 0.5301063
```

In a binomial distribution of 13 trials with p=0.41 find the probability of getting fewer than 7 successes. But P(X<7) is the same as P(X<=6).

```
     # For 13 trials with p=0.41, find P(X<7).
     # But P(X<7) is the same as P(X<=6)
pbinom( 6, 13, 0.41)
```

```
>      # For 13 trials with p=0.41, find P(X<7).
>      # But P(X<7) is the same as P(X<=6)
> pbinom( 6, 13, 0.41)
[1] 0.7476386
```

In a binomial distribution of 9 trials with p=0.573 find the probability of getting between 4 and 7 successes, inclusinve. We note that pbinom(7, 9, 0.573) gives the probability of getting 7 or fewer successes. That is, it is the sum P(X=0) + P(X=1) +P(X=2) + P(X=3) +P(X=4) + P(X=5) + P(X=6) + P(X=7).
But we only want P(X=4) + P(X=5) + P(X=6) + P(X=7). We can do that by finding P(X<=7) and then subtracting P(X<=3).

```
     # For 9 trials with p=0.573, find
     # P(4 <= X <= 7)
pbinom( 7, 9, 0.573 ) - pbinom( 3, 9, 0.573 )
```

```
>      # For 9 trials with p=0.573, find
>      # P(4 <= X <= 7)
> pbinom( 7, 9, 0.573 ) - pbinom( 3, 9, 0.573 )
[1] 0.8161315
```

The next challenge is to use pbinom() to solve P(X=5) for a binomial distribution with 12 trials and p=0.379. The statement pbinom(5, 12, 0.379) solves P(X<=5) not our desired P(X=5). But we can do P(X=5) = P(X<=5) - P(X<=4).

```
     # For 12 trials, p=0.379, find P(X=5).
pbinom( 5, 12, 0.379 ) - pbinom( 4, 12, 0.379 )
```

```
>      # For 12 trials, p=0.379, find P(X=5).
> pbinom( 5, 12, 0.379 ) - pbinom( 4, 12, 0.379 )
[1] 0.2205781
```

This type of problem, finding the value for just a specific number of successes, happend enough that we might want a function that will do this for us. To that end I have created pbinomeq().

```
     # alternatively, ...
source("../pbinomeq.R")
pbinomeq( 5, 12, 0.379)
```

```
> source("../pbinomeq.R")
> pbinomeq( 5, 12, 0.379)
[1] 0.2205781
```

Recall that we solved the issue of finding P(X>=11) for a binomial distribution with 19 trials and p=0.637 by looking at 1 - P(X<=10). The function pbinom() gives us another way to solve such problems. So far a statement such as pbinom(a, b, p) gives us the probability of having a or fewer successes in b trials when the probability of success on a single trial is p.

The new form of the function, pbinom(a, b, p, lower.tail=FALSE) gives the probability of having more than a successes in b trials when the probability of success on a single trial is p. Therefore, we can find P(X>=11) for a binomial distribution with 19 trials and p=0.637 by using the statement pbinom(10, 19, 0.637, lower.tail=FALSE). Note that FALSE must be in all caps.

```
     # solve P(X>=11) with 19 trials and p=0.637
     # two different ways
1 - pbinom( 10, 19, 0.637 )    # the old way
pbinom( 10, 19, 0.637, lower.tail=FALSE)  # the new way
```

```
>      # solve P(X>=11) with 19 trials and p=0.637
>      # two different ways
> 1 - pbinom( 10, 19, 0.637 )    # the old way
[1] 0.7801826
> pbinom( 10, 19, 0.637, lower.tail=FALSE)  # the new way
[1] 0.7801826
```

The use of lower.tail=FALSE takes a bit more typing but it makes more clear the intent of the statement. We will see many more instances of lower.tail=FALSE so it might be a good idea to become familiar with it.

The last items to present here are the fomula for the mean, i.e., the expected value, and the formula for the standard deviation of a binomial distribution. For a binomial distribution having n trials with a probability of success on a single trial being p we have

mean: $E(X) = \mu = n \cdot p$

standard deviation: $\sigma = \sqrt{n \cdot p \cdot (1 - p)}$

For a binomial distribution with 17 trials and a probability of success for a single trial as 0.634 find the mean and standard deviation.

```
     # find the mean and standard deviation for a
     # binomial distribution with 17 trials and p=0.634
mu <- 17*0.634
mu
sigma <- sqrt( 17*0.634*(1-0.634))
sigma
```

```
>      # find the mean and standard deviation for a
>      # binomial distribution with 17 trials and p=0.634
> mu <- 17*0.634
> mu
[1] 10.778
> sigma <- sqrt( 17*0.634*(1-0.634))
> sigma
[1] 1.986139
```